DEPARTMENT OF MATHEMATICS,
D.K.College, Jaleswar.

Semester-V Subject-Mathematics

Paper-Math-Maj-XI

Syllabus:
(Fourier Series, Riemann Stieltjes Integral, Function of Bounded Variables, Rectifiable curves, Calculus of Multi variable Functions,Contraction mapping Theorem,Taylor's theorem in More variables)

UNIT-I(Fourier Series)

1. **Short Answer Type Questions:**

   (a) Define periodic function? Also give an example of two even functions with their period

   (b) State Dirichlet's conditions for the existence of a Fourier series.

   (c) If $f(x)$ is an even function in $(-\pi, \pi)$, What are its Fourier coefficients?

   (d) If $f(x)$ is an odd function in $(-\pi, \pi)$, What are its Fourier coefficients?

   (e) If $f(x)$ is an even function in $(-L, L)$, What are its Fourier coefficients?

   (f) If $f(x)$ is an odd function in $(-L, L)$, What are its Fourier coefficients?

   (g) State the convergence of the Fourier series at a point of discontinuity $x = c$

   (h) Find the constant term $\frac{a_0}{2}$ in the Fourier expansion of $f(x) = x^2$ in $[-\pi, \pi]$.

   (i) State Parseval's identity for a Fourier series in $(-\pi, \pi)$

   (j) Write formula for the half range Fourier sine series and cosine series in $(0, L)$

   (k) State and prove Bessele's Inequallity

   (l) State and prove Parseval's Formula

2. **Long Answer type questions**

   (a) Expand f in a Fourier series in the interval $[-\pi, \pi]$, If $f(x) = \begin{cases} 1 & for - \pi < x \leq 0 \\ -2 & for\ 0 < x \leq \pi \end{cases}$

(b) Express Fourier series expansion of $f(x)$ defined by $f(x) = \begin{cases} -\pi & for -\pi < x \leq 0 \\ x & for\ 0 < x \leq \pi \end{cases}$.

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} = \frac{\pi^2}{8}$

(c) Find Fourier series for $f(x)$ defined by $f(x) = \begin{cases} x & for\ 0 < x \leq 1 \\ 1-x & for\ 1 < x \leq 2. \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} = \frac{\pi^2}{8}$

(d) Find Fourier series for $f(x)$ defined by $f(x) = \begin{cases} 0 & for\ -\pi \leq x \leq 0 \\ 1 & for\ 0 \leq x \leq \pi \end{cases}$

(e) If f is bounded and integrable on $[-\pi, \pi]$ and a_n, b_n are its Fourier coefficients, Prove that the series $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is convergence.

(f) Find Fourier series for $f(x) = x \sin x$ in the interval $(-\pi, \pi)$.
Also show that $\frac{\pi-1}{4} = \frac{1}{1\times3} - \frac{1}{3\times5} + \frac{1}{5\times7} - \frac{1}{7\times9} + ...$

(g) If a function ϕ is bounded and integrable on the interval $[a, b]$, then show that $A_n = \int_a^b \phi \cos nx \longrightarrow 0$ and $B_n = \int_a^b \phi \sin nx \longrightarrow 0$

(h) **Best Approximation Theorem** If $\{\phi_1, \phi_2, ...\}$ be Orthonormal on I and $f \in L^2(I)$. Define two sequence of functions $\{s_n\}$ and $\{t_n\}$ on I by $s_n(x) = \sum_{k=0}^{k=n} b_k\phi_k(x)$ and $t_n(x) = \sum_{k=0}^{k=n} c_k\phi_k(x)$ respectively, where $c_k = (f, \phi_k)$ for $k = 0, 1, 2...$ and $b_0, b_1, b_2...$ are arbitrary complex numbers. Then Prove that: for each n, we have $||f - t_n|| \leq ||f - s_n||$, moreover, equality holds if $b_k = c_k, k = 0, 1, 2....$

(i) State and prove The Riesz-Fischer Theorem

UNIT-II
(Function of Bounded Variation and Riemann-Stieltjes Integral)

1. **Short answer type questions**

(a) Define bounded variation of a function?

(b) A function is of bounded variation iff it can be expressed as the difference........?

(c) A curve $V : [a, b] \longrightarrow \mathbb{R}^n$ is rectifiable, if its length is ......?

(d) If f is constant function, then its bounded variation on any closed bounded interval is .....?

(e) $f : [0, 1] \longrightarrow \mathbb{R}$ such that $f(x) = x$, then find $V(f, P)$, where partition $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

(f) Give an example of a function that continuity is not necessary for bounded variation.

(g) Give an example of a bounded function that continuity is not sufficient for bounded variation?

(h) Give an example of a function, that boundedness of its derivative is sufficient for function is bounded variation but not necessary?

(i) If f is of bounded variation on $[a, b]$, for all partition of $[a, b]$.Then f is bounded on $[a, b]$.

(j) Show that $f(x) = \begin{cases} x^2 \cos(\frac{1}{x}) & for\ x \neq 0 \\ 0 & for\ x = 0 \end{cases}$

(k) Show that the unit circle $\gamma(t) = e^{it} = \cos t + i \sin t, t \in [0, 2\pi]$ is rectifiable?

(l) Show that $f(x) = x \sin \frac{1}{x}$ is not rectifiable on $[0, 1]$?

(m) what is the length of a rectifiable curve?

(n) Define Partition of an interval?

(o) Define norm of a partition? If $I = [1, 13]$ be a closed and bounded interval in R, where $P = \{1, 2, 5, 9, 12, 13\}$, be any partition of I. Then find $||P||$.

(p) Define refinement of a partition. If $[2, 8]$ be closed and bounded interval in $\mathbb{R}$, $P_1 = (2, 5, 7, 8)$, $P_2 = (2, 4, 6, 7, 8)$. Then find the common refinement of P_1 and P_2.

(q) If P is a refinement of Q, then which one is correct
(i) $L(P, f, \alpha) \leq L(Q, f, \alpha)$ (ii) $U(P, f, \alpha) \leq U(Q, f, \alpha)$
(iii) $L(Q, f, \alpha) \leq L(P, f, \alpha)$ (iv) $L(P, f, \alpha) \leq U(Q, f, \alpha)$

(r) The lower Riemann-Stieltjes integral can not exceed the upper Riemann-Stieltjes integral. i.e $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$

(s) If f is monotonic on $[a, b]$ and if α is continuous on $[a, b]$ and monotonic increasing. Then prove that f is RS-integral?

(t) Find $\int_0^\pi x d(\sin x)$

(u) Find $\int_0^2 [x] d(x^2)$

2. **Long Answer type questions**

(i) Show that a continuous function $f(x) = \begin{cases} x\sin(\frac{\pi}{x}) & for\ 0 \leq x \leq 1 \\ 0 & for\ x \leq 0 \end{cases}$ may not be a function of bounded variation?

(ii) If f is monotonic on $[a, b]$, then show that f is of bounded variation on $[a, b]$.

(iii) If f is continuous on $[a, b]$ and if f' exist and bounded on (a, b). Then prove that f is of bounded variation on $[a, b]$.

(iv) If f and g are two function of bounded variation, then show that their sum(f+g) is also function odf bounded variation?

(v) Let f be defined on $[a, b]$. Prove that: f is of bounded variation on $[a, b]$ iff f can be expressed as the difference of two increasing functions.

(vi) If $f(x) = \begin{cases} x\cos(\frac{\pi}{2x}) & for\ x \neq 0 \\ 0 & for\ x = 0 \end{cases}$. Then show that $f(x)$ is continuous on $[0, 1]$ but not a function of bounded variation?

(vii) If f is continuous on $[a, b]$ and α is function of bounded variation. Then prove that f is RS-integral.

(viii) Let f and α be bounded functions on $[a, b]$ and α be monotonic increasing on $[a, b]$. Then show that the function f is integrable with respect to α on $[a, b]$ if and only if for every $\epsilon > 0$, there exist a partition P on $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

(ix) State and prove First Mean Value theorem for Riemann Stieltjes integrable.

UNIT-III(Multi-Variable Functions)

1. **Short answer type questions**

(a) Let $f(x, y) = \sqrt{y+1} + ln(x^2 - y)$. Find $f(e, 0)$ Sketch the natural domain of f.

(b) Let $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$. Find $f(0, \frac{1}{2}, -\frac{1}{2})$ and the natural domain of f.

(c) In each part ,describe the graph of the function in xyz-coordinate system.
(i) $f(x,y) = 1 - x - \frac{y}{2}$ (ii) $f(x,y) = \sqrt{1 - x^2 - y^2}$ (iii) $f(x,y) = -\sqrt{x^2 - y^2}$

(d) Evaluate: $\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2}$

(e) Find $\lim_{(x,y)\to(0,0)}(x^2 + y^2)\log(x^2 + y^2)$

(f) Show that the $\lim_{(x,y)\to(0,0)} f(x,y)$ is does not exist, where $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & for\ (x,y) \neq (0,0) \\ 0, & for\ (x,y) = (0,0) \end{cases}$

(g) Show that the $\lim_{(x,y)\to(0,0)} f(x,y)$ is does not exist, where $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & for\ (x,y) \neq (0,0) \\ 0, & for\ (x,y) = (0,0) \end{cases}$

(h) Evaluate: $\lim_{(x,y)\to(1,2)} \frac{(x^2-1)(y^2-4)}{(x-1)(y-2)}$ (UG-Math-2024)

(i) Evaluate: $\lim_{(x,y)\to(0,0)}(1 - \frac{\sin(x^2+y^2)}{x^2+y^2})$ (UG-Math-2024)

(j) If $f(x,y) = \begin{cases} y\sin\frac{1}{x}, & for\ x \neq 0 \\ K, & for\ x = 0 \end{cases}$ is continuous at origin, then find k (UG-Math-2024)

(k) Find the directional derivative of $f(x,y,z) = x^4 + y^4 + z^4$ at $(1,-2,1)$ in the direction of $\hat{\imath} + \hat{\jmath} + \hat{k}$

(l) Find unit normal vector to the surface $xy^3 = z + 2$, at $(1,1,-1)$

(m) Find the greatest value of the directional derivative of the function $xy + 2yz + 3xz$ at the point $(1,1,1)$

(n) If $z = e^{xy}, x = 2u + v, y = \frac{u}{v}$, then find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using chain rule.

2. **Long Answer type questions**

(a) $f(x,y) = 2x + 3y$, Show that $\lim_{(x,y)\to(1,1)} f(x,y) = 5$ by ϵ-δ definition method

(b) Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ by ϵ-δ definition method,
where $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, & for\ (x.y) \neq 0 \\ 0, & for\ (x.y) = 0 \end{cases}$

(c) Prove that the $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ where $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & for\ (x,y) \neq (0,0) \\ 0, & for\ (x,y) = (0,0) \end{cases}$

(d) Calculate the function f given by $f(x,y) = \begin{cases} xy(\frac{x^2-y^2}{x^2+y^2}), & for\ (x,y) \neq (0,0) \\ 0, & for\ (x,y) = (0,0) \end{cases}$ for continuity and differentiable and possesses partial derivative.

(e) If $z = xy + f(x^2 + y^2)$, show that $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y^2 - x^2$

(f) Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & for\ (x,y) \neq (0,0) \\ 0, & for\ (x,y) = (0,0) \end{cases}$,
then compute $f_x, f_y, f_x(0,0), f_y(0,0)$.

(g) If $u = (1 - 2xy + y^2)^{\frac{-1}{2}}$, Prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$

(h) $f(x,y) = \begin{cases} x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}, & for\ (x,y) \neq (0,0) \\ 0, & for\ (x,y) = (0,0) \end{cases}$
Show that $f_{xy} \neq f_{yx}$ at $(0,0)$

(i) Show that $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$

(j) Prove that Every differentiable function is continuous. Explain by counter example that converse is not true?

(k) A function $f : R^2 \longrightarrow R$, which is differentiable at a point possesses the first order partial derivatives at that point but converse is not true ,explain by counter example.

(l) Let f be a real valued function defined on neighbourhood of (a, b). If f_x is continuous at (a, b) and f_y exists at (a,b), then prove that f is differentiable at (a, b).

(m) State and prove Contraction Mapping Theorem.

(n) Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$, by using Taylor's expansion